Journal of Applied Mechanics and Technical Physics, Vol. 35, No. 5, 1994

GENERATION OF WAVES IN A LAYERED MEDIUM CONTAINING A LOCAL DEFECT

A. A. Lyapin

UDC 539.3:534.1

We propose to investigate the boundary-value problem of the generation of elastic waves in a layered half-space with a surface indentation or an interior void in the presence of shear strain. Oscillations are excited by a load on the exposed (top) surface of the medium. In the investigation we use the method of boundary integral equations, based on the dynamic reciprocity theorem. We formulate appropriate integral representations for the wave field in the medium in terms of the distribution of displacements of defect points. We analyze the resulting boundary integral equations numerically as a function of the relations between the physical characteristics of the medium. The results are targeted for practical applications in flaw detection, earthquake-resistant construction, and vibration sounding of the earth.

Investigations of the behavior of elastic media in the form of a layer or half-plane containing voids of arbitrary configuration or surface indentations have been reported in a great many papers to date [1-5].

The objective of the present study is to investigate the behavior of elastic bodies containing such defects of arbitrary configuration for media with a plane-parallel layered structure (stacked layers or a multilayer half-plane).

1. We consider (without sacrificing generality) a two-layer half-plane with a defect in the form of a void or surface indentation contained wholly within a layer. The geometry of the region is described in Cartesian coordinates (x, y, z) by the relations

$$D_1 = \{x > 0;\ y \in (-\infty, +\infty)\} \text{ (half-plane)};$$
$$D_2 = \{x \in (-h, 0);\ y \in (-\infty, +\infty);\ (x, y) \notin \Omega\} \text{ (layer with defect)}$$

where Ω is a simply connected, compact domain in D_2, bounded by a piecewise-smooth curve γ.

Steady-state harmonic oscillations of frequency ω are generated by a shear load on the surface of the layer, for example, a point force

$$\tau_{xz}|_{x=-h} = \delta(y - y_*)\exp(-i\omega t).$$

The rest of the boundary (including the surface of the defect) is assumed to be stress-free. The layer is rigidly attached to an underlying half-plane, and the requirement of continuity of the displacements $w(x, y)$ and stresses $\tau_{xz}(x, y)$ in transition across the interface $x = 0$ is satisfied. The shear displacements $w_j(x, y)$ ($j = 1$ for the half-plane, $j = 2$ for the layer, and the media are characterized by the parameters V_{sj} and μ_j, which represent the wave velocity and shear moduli, respectively) satisfy the Helmholtz equation, each in its own domain.

Invoking the limiting absorption principle, we can represent the function describing the displacement field in the half-plane by a Fourier integral around a contour Γ in the plane of the complex parameter α [6]:

$$w_1(x, y) = \frac{\zeta}{2\pi}\int_\Gamma \exp(-i\alpha y)\, T(\alpha)\, P(x, \alpha)\, d\alpha,$$
$$P(x, \alpha) = -\exp(-\sigma_1 x)/\sigma_1,\ \sigma_j = \sqrt{\alpha^2 - \theta_j^2},\ \theta_j = \omega h/V_{sj},\ \zeta = \mu_2/\mu_1,\quad j = 1, 2. \tag{1.1}$$

Here all linear parameters, including the displacement functions, are normalized to the thickness of the layer h, and the stress functions are normalized to the shear modulus μ_2. We omit the time factor $\exp(-i\omega t)$ everywhere; $T(\alpha) = F_y[\tau(y)]$ is the

Rostov-on-Don. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 5, pp. 87-91, September-October, 1994. Original article submitted November 23, 1993.

0021-8944/94/3505-0722$12.50

Fourier transform of the contact stresses at the interface. When the underlying medium is a multilayer half-plane or stacked layers, the form of the function $P(x, \alpha)$ changes, but the only difficulties this creates for the investigator are numerical.

To determine the displacements in the layer with the defect, we use the direct formulation of the method of boundary integral equations, based on the dynamic reciprocity theorem [7]. Of all the fundamental solutions used within the scope of the present study, the one described below is the most efficient. We introduce the Green's function corresponding to the effect of a point shear force applied at the point (x_0, y_0) of the layer without any stresses on its faces $x = 0$ and $x = -1$:

$$w_*(x, y, x_0, y_0) = \frac{i}{4}H_0^{(1)}(\theta_2 R) + \frac{i}{4}H_0^{(1)}(\theta_2 R_0) + \frac{i}{4}H_0^{(1)}(\theta_2 R_{-1})$$

$$+ \frac{1}{2\pi}\int_\Gamma \exp(-i\alpha y)\, G(x, \alpha, x_0, y_0)d\alpha,\ R = \sqrt{(x - x_0)^2 + (y - y_0)^2},$$

$$R_0 = \sqrt{(x + x_0)^2 + (y - y_0)^2},\ R_{-1} = \sqrt{(x + x_0 + 2)^2 + (y - y_0)^2},$$

$$G(x, \alpha, x_0, y_0) = \exp(i\alpha y_0)\,[\exp(\sigma_2(x + x_0))(\exp(-2\sigma_2) + \exp(-2\sigma_2(1 + x)))$$

$$+ \exp(-\sigma_2(2 + x + x_0))(\exp(-2\sigma_2) + \exp(2\sigma_2 x))\,](\sigma_2(1 - \exp(-2\sigma_2)))^{-1}.$$

Here the components outside the integral sign represent the sum of the direct field of the source plus waves reflected once from the boundaries of the layer, and the term described by the contour integral represents multiply reflected waves. This form of the function w_* lends itself to fast computation, regardless of the relations between the parameters (x, y) and (x_0, y_0). We assume that the external forces applied to the layer with the defect correspond to the boundary conditions of the basic boundary-value problem and the interface matching condition:

$$\tau_{xz}|_{x = 0} = \tau(y),\ \tau_{xz}|_{x = -1} = \delta(y - y_*),\ \tau_{nz}|_{(x,y)\in\gamma} = 0.$$

From the reciprocity relations we readily obtain an analog of the Kirchhoff equation for determining the displacements of points of the layer, including the boundaries $x = 0$ and $x = -1$:

$$w_2(x_0, y_0) = -\int_\gamma q_*(x, y, x_0, y_0)\, w_2(x, y)ds$$

$$+ \int_{-\infty}^{+\infty} \tau(y) w_*(0, y, x_0, y_0)dy + w_*(-1, y_*, x_0, y_0), \tag{1.2}$$

$$q_*(x, y, x_0, y_0) = \partial w_*(x, y, x_0, y_0)/\partial n(x, y).$$

In the derivation of the equations significant use is made of the property of the fundamental solution

$$\partial w_*(x, y, x_0, y_0)/\partial x = 0 \text{ at } x = 0 \text{ and } x = -1,$$

along with the relation for $(x_0, y_0) \in \gamma$

$$\lim_{y \to \pm\infty} \int_{-h}^{0} [w_*(x, y, x_0, y_0)\partial w_2/\partial y - \partial w_*/\partial y(x, y, x_0, y_0)w_2]dx = 0,$$

which is a consequence of the asymptotic behavior of the functions w_* and w_2 as a finite set of propagating modes and cylindrical waves exhibiting power-law decay with increasing distance from the source.

Letting (x_0, y_0) tend toward the boundary in Eq. (1.2) and taking into account the discontinuity of the integral on its right-hand side, we obtain the boundary integral equation

$$\xi(x_0, y_0)w_2(x_0,y_0) + \int_\gamma q_*(x, y, x_0, y_0)w_2(x, y)ds$$

$$= \int_{+\infty}^{-\infty} \tau(y)w_*(0, y, x_0, y_0)dy + w_*(-1, y_*, x_0,y_0),\ (x_0, y_0) \in\gamma. \tag{1.3}$$

Here $\xi(x_0, y_0) = 0.5$ for regular points of the boundary γ, $\xi(x_0, y_0) = \beta/(2\pi)$ for points of the boundary $x_0 - 1$ where it changes discontinuously, β is the interior angle corresponding to the discontinuity, and $\xi(x_0, y_0) = \beta/\pi$ for points of the boundary γ at $x_0 = -1$ (surface indentation).

To determine $w_2(x_0, y_0)$ on γ, we eliminate the function $\tau(y)$ from Eq. (1.3) by virtue of the equality of the displacements in transition from D_1 to D_2 across the interface $x = 0$. Accordingly, we use reexpansion equations [8] to write the Green's function in Fourier integral form:

$$w_* = \frac{1}{2\pi}\int \exp[i\alpha(y_0 - y)]W_*(x, \alpha, x_0)d\alpha,$$

$$W_*(x, \alpha, x_0) = \frac{\mathrm{ch}[\sigma_2 \max(x, x_0)]\mathrm{ch}[\sigma_2(1 + \min(x, x_0))]}{\sigma_2 \mathrm{sh}\sigma_2}.$$

We can therefore write the function q_* as

$$q_* = \frac{1}{2\pi}\int_\Gamma \exp[i\alpha(y_0 - y)]Q_*(x, \alpha, x_0)d\alpha.$$

Next, applying the Fourier transform in the coordinate y_0 to Eq. (1.2) and making use of the relations

$$F_{y_0}\left[\int_{+\infty}^{-\infty}\tau(y)w_*(0, y, x_0, y_0)dy\right] = W_*(0, -\alpha, x_0)T(\alpha),$$

$$F_{y_0}[-\int_\gamma q_*(x, y, x_0, y_0)w_2(x, y)ds] = -\int_\gamma Q_*(x, -\alpha, x_0)\exp(i\alpha y)w_2(x, y)ds$$

we obtain the following expression for the Fourier transforms of the contact stresses $T(\alpha)$:

$$T(\alpha) = \int_\gamma (Q_*(x, -\alpha, 0)w_2(x, y)\exp(i\alpha y)ds W_*(-1, -\alpha, 0)\exp(i\alpha y_*))/\Delta(\alpha),$$

$$\Delta(\alpha) = W_*(0, \alpha, 0) - \zeta P(0, -\alpha).$$

As a result, to determine the boundary distribution of the displacements on a defect in a two-layer medium, we have the integral equation

$$\xi(x_0, y_0)w(x_0, y_0) + \int_\gamma [q_*(x, y, x_0, y_0) + q_*^+(x, y, x_0, y_0)]w_2(x, y)ds$$

$$= w_*(-1, y_*, x_0, y_0) + w_*^+(-1, y_*, x_0, y_0), \ (x_0, y_0) \in \gamma,$$

$$q_*^+(x, y, x_0, y_0) = -\frac{1}{2\pi}\int_\Gamma \exp[i\alpha(y_0 - y)]W_*(0, \alpha, x_0)Q_*(x, \alpha, 0)/\Delta(\alpha), \tag{1.4}$$

$$w_*^+(x, y, x_0, y_0) = -\frac{1}{2\pi}\int_\Gamma \exp[i\alpha(y_0 - y)]W_*(0, \alpha, x_0)W_*(x, \alpha, 0)/\Delta(\alpha).$$

Consequently, the fundamental solution

$$w_*(x, y, x_0, y_0) + w_*^+(x, y, x_0, y_0) \tag{1.5}$$

is a special type of solution that satisfies zero-valued boundary conditions on the exposed (top) surface of the layer and the conditions of rigid attachment to the underlying medium on the lower surface. Note that the zeros of the function $\Delta(\alpha)$ correspond to the wave numbers of Love waves in a two-layer half-plane (without a defect) and are bypassed by the contour Γ in the complex plane of the parameter α: positive downward, and negative upward. If the layer and the half-plane have identical elastic parameters, the fundamental solution (1.5) degenerates into the Green's function for a half-plane:

$$\frac{i}{4}H_0^{(1)}(\theta_2 R) + \frac{i}{4}H_0^{(1)}(\theta_2 R_{-1}).$$

The equivalence of the boundary integral equation (1.5) and the original boundary-value problem is easily demonstrated on the basis of the representation of the fundamental solution.

Once the function $w_2(x, y)$ has been determined on γ, the wave field in the entire domain, including the boundary, is reconstructed from Eqs. (1.1) and (1.2) with allowance for Eq. (1.4).

The procedure described here extends directly to problems of oscillations of a layered medium with a defect in the planar case.

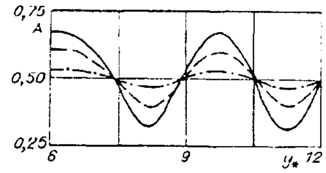

Fig. 1

2. As an example illustrating the calculation of the characteristics of the wave motion of a layer with a surface indentation, we consider the behavior of the amplitude of a surface wave in the far field of the source

$$| y_0 - y_* | \gg 1$$

as a function of the shape of the indentation. The surface of the indentation is characterized by the parameter $\varkappa$ corresponding to its parametrization:

$$y = \varkappa \cos\vartheta,\ x = -0.5 - 0.5\cos^2\vartheta,\ \vartheta \in [0, \pi],\ \varkappa \in (0; 0.5].$$

The value of the parameter $\varkappa = 0$ correspond to a vertical surface-breaking crack of depth 0.5.

The amplitude of the surface wave at frequencies $\theta_2 < \pi$ can be expressed by the relation

$$A = \frac{i}{2\theta_2}[\exp(-i\theta_2 y_*) + i\theta_2 \int_\gamma \exp(-i\theta_2 y)\cos(\mathrm{n}, \mathrm{e}_y) w(x, y) ds].$$

The first term characterizes the surface wave amplitude in the defect-free layer. It is evident from this equation that the surface wave energy is least influenced by defects in the form of planar cracks oriented parallel to the boundaries of the layer. For vertical cracks the deviation from the corresponding curve for the defect-free layer is proportional to the opening of the crack. Figure 1 shows the amplitude A as a function of the parameter y_* for surface indentations of various shapes: $\varkappa = 0.5$ (solid curve); $\varkappa = 0.1$ (dashed curve); $\varkappa = 0.02$ (dot-dash curve). Beginning with a certain value of y_* of the order of the shear wavelength in the layer, the dependence of A on y_* exhibits a periodic behavior. The modulus of the amplitude A oscillates about the corresponding value for the defect-free layer, deviating from it by an amount that decreases as the parameter $\varkappa$ decreases in the range (0, 0.5). We note that these functional relations are typical for $\theta_2 < \pi$, i.e., for relatively thin layers and small defect dimensions in comparison with the shear wavelength in the medium.

REFERENCES

1. E. V. Zakharov and I. V. Il'in, "Method for the calculation of electromagnetic fields in a plane-parallel layered medium with local inhomogeneities," in: Computational Methods and Programming [in Russian], No. 16, Izd. MGU, Moscow (1971).
2. R. A. Roberts, "Elastodynamic scattering by a surface-breaking void," J. Acoust. Soc. Am., **85**, No. 2 (1989).
3. A. O. Vatul'yan and A. Ya. Katsevich, Vibrations of an orthotropic elastic layer containing a void," Prikl. Mekh. Tekh. Fiz., No. 1 (1991).
4. L. A. Alekseeva, "Boundary integral equations for an elastic half-plane with a hole subjected to dynamic edge loading," Izv. Akad. Nauk Kaz. SSR Ser. Fiz. Mat., No. 3 (1987).
5. D. E. Budreck and J. D. Achenbach, "Three-dimensional elastic wave scattering by surface-breaking cracks," J. Acoust. Soc. Am., **86**, No. 1 (1989).
6. I. I. Vorovich and V. A. Babeshko, Dynamic Mixed Problems in the Theory of Elasticity for Nonclassical Domains [in Russian], Nauka, Moscow (1979).
7. P. K. Banerjee and R. Butterfield, Method of Boundary Elements in Engineering Science, McGraw-Hill, New York (1982).
8. P. M. Morse and H. Feshbach, Methods of Theoretical Physics, Vol. 2, McGraw-Hill, New York (1953).